ISSN:2319-2607 IISUniv.J.Sc.Tech.Vol. 7&8(1), 61-67 (2018 & 19)
RESEARCH ARTICLE

Caputo Extended Derivative Operator Using Confluent Hypergeometric Function

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Abstract
Extensions of Fractional derivative operator was done by many researchers. In this sequal, the paper deals with an extended Caputo fractional derivative operator of some fundamental functions derived by an extension of beta funtion in the kernal. We developed an extended Caputo fractional derivative operator with the help of following papers Agrawal et al.,(2015), (Agrawal and Choi 2016), Kiymaz et al.,(2016), Rahman et al.,(2018), Rahman et al.,(2019) with their properties.

Keywords: Extended Beta Function, Fractional Derivative operator of Caputo, Confluent Hypergeometric function, fractional derivative, Mellin transformation. AMS Subject Classification: 26A33, 33B15, 33C05, 65D20, 33E20, 91B25, 26A33.

Introduction

Many extensions on special function and fractional calculus were done. Because of its importance in various fields, many authors develop generalized incomplete form of the fractional derivative operator along with confluent hypergeometric function.

The basic confluent hypergeometric function is defined as:

$$ {}_1F_1(\Upsilon;\upsilon;z) := \sum_{n=0}^{\infty} \frac{(\Upsilon)_n}{(\upsilon)_n}\frac{z^n}{n!}, (|z|<1), \qquad (1)$$

$$(\Upsilon,\upsilon \in C \quad and \quad \upsilon \neq 0,-1,-2,-2....).$$

The Appell series is defined by:

$$F_1(\Upsilon,\upsilon,\tau;\varphi;x,y)$$

$$:= \sum_{m,n=0}^{\infty} \frac{(\Upsilon)_{n+m}(\upsilon)_m(\tau)_n}{(\varphi)_{m+n}}\frac{x^n y^m}{n!m!}, \qquad (2)$$

$$\forall \quad \Upsilon,\upsilon,\tau;\varphi \in C$$

and

$$\varphi \neq 0,-1,-2,...., |x|<1, |y|<1.$$

Its integral representation is given by:

$$F_1(\Upsilon,\upsilon,\tau;\varphi;x,y) := \frac{\Gamma(\varphi)}{\Gamma(\Upsilon)\Gamma(\varphi-\Upsilon)}\times$$

$$\int_0^1 r^{\Upsilon-1}(1-r)^{\varphi-\Upsilon-1}(1-xr)^{-\upsilon}(1-yr)^{-\tau}dr. \qquad (3)$$

Extended beta function developed by Srivastava *et al.* (2014) and Gauss hypergeometric function defined as:

$$B_h^{\zeta,\eta;k,u}(x,y)$$

$$:= \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\zeta;\eta;\frac{-h}{t^k(1-t)^u}\right)dt, \qquad (4)$$

$$min(Re(\zeta),Re(\eta),Re(k),Re(u))>0,$$

$$Re(x)>-Re(k\zeta), Re(h)\geq 0,$$

$$Re(y)>,-Re(u\zeta),$$

and

$$F_h^{(\zeta,\eta;k,u)}(a,b;c;z)$$

$$:= \sum_{n=0}^{\infty}(a)_n \frac{B_h^{(\zeta,\eta;k,u)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \qquad (5)$$

$$|z|<1, min(Re(\zeta),Re(\eta),Re(k),Re(u))>0,$$

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$$Re(c) > Re(b) > 0, Re(h) \geq 0.$$

Which is reducible to the generalized Beta type function defined by (Parmar, 2013) when $k = u$. He studied some fundamental properties and characteristics of this generalized Beta type function as:

$$B_h{}^{\zeta,\eta;u}(x,y)$$

$$:= \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\zeta;\eta;\frac{-h}{t^u(1-t)^u}\right)dt, \quad (6)$$

where $Re(h) \geq 0, min(Re(x), Re(y), Re(\zeta), Re(\eta), Re(u)) > 0.$

Equation (6) is reduced into the special case, for

- $u = 1$, $B_h{}^{\zeta,\eta;u}(x,y)$ reduced into $B_h{}^{\zeta,\eta}(x,y)$.
- $u = 1$ and $\zeta = \eta$, $B_h{}^{\zeta,\eta;u}(x,y)$ reduced into $B_h(x,y)$.
- $u = 1$, $\zeta = \eta$ and $h = 0$, $B_h{}^{\zeta,\eta;u}(x,y)$ reduced into $B(x,y)$.

For details see, Chaudhry *et al.* (1997), Chaudhry *et al.* (2004), Özergin *et al.* (2011), Özergin (2011), Cho and Srivastava (2014), Luo *et al.* (2014).

Gauss hypergeometric function which was defined by Parmar, (2013) as: $F_h^{(\zeta,\eta;u)}(a, b; c; z)$

$$:= \sum_{n=0}^{\infty}(a)_n \frac{B_h{}^{(\zeta,\eta;u)}(b+n, c-b)}{B(b, c-b)}\frac{z^n}{n!}, \quad (7)$$

where $|z| < 1, Re(h) \geq 0, \min(Re(\zeta), Re(\eta), Re(u)) > 0, Re(c) > Re(b) > 0, Re(a) > 0.$

Again, extended Gauss hypergeometric function's extension was defined by Praveen Agrawal *et al.* (2015) by using extended beta function which was developed by Srivastava such as:

$$F_{h;k,u}(a, b; c; z; \Theta) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \times$$

$$\frac{B_h{}^{\zeta,\eta;k,u}(b+n, c-b+\Theta)}{B(b+n, c-b+\Theta)}\frac{z^n}{n!}, \quad (8)$$

where, $(\Theta < Re(b) < Re(c); |z| < 1, Re(k) > 0, Re(u) > 0), Re(h) \geq 0, \Theta \in N.$

Classical Caputo fractional derivative operator which is defined as:

$$C_\varrho^\varpi(f(z)) = \frac{1}{\Gamma(m-\varpi)}\int_0^\varrho (\varrho - t)^{m-\varpi-1}\frac{d^m}{dt^m}f(t)dt, \quad (9)$$

for

$$m - 1 < R(\varpi) < m \quad where \quad m = 1, 2, 3........$$

There were several extensions derived of the equation Classical Caputo fractional derivative by authors eg: Marchado *et al.* (2011), Kiymaz *et. al.*(2016), Rahman *et. al.*(2018), Kilbas *et. al.* (2006).

Caputo Extended Fractional Derivative

In this section, we introduce and investigate new extension with the help of the Extended beta function as Confluent hypergeometric function in its kernal, defined as:

$$C_\varrho^\varpi(f(\varrho), h, \Upsilon, \upsilon) = \frac{1}{\Gamma(m-\varpi)}\int_0^\varrho (\varrho - t)^{m-\varpi-1} \times$$

$${}_1F_1\left(\Upsilon;\upsilon;\frac{-h\varrho^{2k}}{t^k(\varrho-t)^k}\right)\frac{d^m}{dt^m}f(t)dt, \quad (10)$$

where $m - 1 < R(\varpi) < m$ where $R(h) > 0, \quad Re(\Upsilon) > 0,$

$$Re(\upsilon) > 0, Re(\Upsilon) > Re(\upsilon), m = 1, 2, 3.. \quad (11)$$

If $\Upsilon = \upsilon$, then equation (10) is converted into Caputo extended fractional derivative operator as:

$$C_\varrho^\varpi(f(\varrho), h) = \frac{1}{\Gamma(m-\varpi)}\int_0^\varrho (\varrho - t)^{m-\varpi-1} \times$$

$$exp\left(\frac{-h\varrho^{2k}}{t^k(\varrho-t)^k}\right)\frac{d^m}{dt^m}f(t)dt, \quad (12)$$

where, $m-1 < R(\varpi) < m$,

where $R(h) > 0, m = 1, 2, 3$........

When h=0 in equation (12) it is reduced into the Classical form of Caputo derivative operator.

Here, we are dealing with various applications of new operator.

Theorem 2.1. *The leading formula hold true*

$$C_{\varrho}^{\varpi}(\varrho^n, h, \Upsilon, \upsilon) = \frac{\Gamma(n+1)}{\Gamma(n-\varpi+1)}\times \frac{{B_h}^{\Upsilon,\upsilon,k}(n-m+1, m-\varpi)}{B(n-m+1,m-\varpi)}\varrho^{n-\varpi}, \Re(\varpi) > 0, \quad (13)$$

where $m-1 < \Re(\varpi) < m$ *and* $\Re(\varpi) < \Re(n)$.

Proof. Applying definition, which is defined by equation (10) on the L.H.S, we get

$$C_{\varrho}^{\varpi}((\varrho)^n, h, \Upsilon, \upsilon)$$

$$:= \frac{1}{\Gamma(m-\varpi)}[n(n-1....(n-m+1))]\times \int_0^{\varrho}(\varrho-t)^{m-\varpi-1}{}_1F_1\Big(\Upsilon;\upsilon;\frac{-h\varrho^{2k}}{t^k(\varrho-t)^k}\Big)(t)^{n-m}dt, \quad (14)$$

$$:= \frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m-\varpi)}\int_0^{\varrho}(\varrho-t)^{m-\varpi-1} {}_1F_1\Big(\Upsilon;\upsilon;\frac{-h\varrho^{2k}}{t^k(\varrho-t)^k}\Big)(t)^{n-m}dt, \quad (15)$$

substituting $t = u\varrho$ in the equation (15), we have

$$C_{\varrho}^{\varpi}((\varrho)^n, h, \Upsilon, \upsilon) = \frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m-\varpi)}\varrho^{n-\varphi}\times \int_0^1 (u)^{n-m}(1-u)^{m-\varphi-1}{}_1F_1\Big(\Upsilon;\upsilon;\frac{-h}{u^k(1-u)^k}\Big)du, \quad (16)$$

by applying the definition of the extended beta function defined by equation (6), we get

$$C_{\varrho}^{\varpi}((\varrho)^n, h, \Upsilon, \upsilon) = \frac{\Gamma(n+1)}{\Gamma(n-\varphi+1)} \frac{{B_h}^{\Upsilon,\upsilon,k}(n-m+1, m-\varpi)}{B(n-m+1,m-\varpi)}\varrho^{n-\varpi}, \quad (17)$$

hence, equation (17) is the desired result.

Theorem 2.2. *If the function* $f(\varrho)$ *is analytic on the disk* $|z| < r$ *for some* $r \in \Re^+$ *and* $m-1 < \Re(\varpi) < m$, *where the power series expansion of a function is given by* $f(\varrho) = \sum_{n=0}^{\infty} a_n\varrho^n$. *Then*

$$C_{\varrho}^{\varpi}(f(\varrho), h, \Upsilon, \upsilon) = \sum_{n=0}^{\infty} a_n C_{\varrho}^{\varpi}((\varrho)^n, h, \Upsilon, \upsilon). \quad (18)$$

Proof. Using the above mentioned series expansion method of function $f(\varrho)$ in the equation (10), we get

$$C_{\varrho}^{\varpi}(f(\varrho), h, \Upsilon, \upsilon) = \frac{1}{\Gamma(m-\varpi)}\int_0^{\varrho}(\varrho-t)^{m-\varpi-1}\times {}_1F_1\Big(\Upsilon;\upsilon;\frac{-h\varrho^{2k}}{t^k(\varrho-t)^k}\Big)\sum_{n=0}^{\infty}a_n\frac{d^m}{dt^m}(t)^n dt, \quad (19)$$

as the series is convergent uniformly and the integrand is also an absolute convergent. Hence after interchanging the order of summation and integration, we get as below:

$$C_{\varrho}^{\varpi}(f(\varrho), h, \Upsilon, \upsilon)$$

$$= \sum_{n=0}^{\infty} a_n \frac{1}{\Gamma(m-\varpi)}\int_0^{\varrho}(\varrho-t)^{m-\varpi-1} \quad (20)$$

$${}_1F_1\Big(\Upsilon;\upsilon;\frac{-h\varrho^{2k}}{t^k(\varrho-t)^k}\Big)\frac{d^m}{dt^m}(t)^n dt,$$

$$= \sum_{n=0}^{\infty} a_n C_{\varrho}^{\varpi}((\varrho)^n, h, \Upsilon, \upsilon). \quad (21)$$

This is the required result.

Theorem 2.3. *If the function* $f(\varrho)$ *is analytic on the disk* $|z| < r$ *for some* $r \in \Re^+$ *and assume* $m-1 < \Re(\varpi) < m$, *where its power series expansion given by* $f(\varrho) = \sum_{n=0}^{\infty} a_n\varrho^n$. *Then*

$$C_{\varrho}^{\varpi}(\varrho^{n-1}f(\varrho), h, \Upsilon, \upsilon) = \frac{\Gamma(n)}{\Gamma(n-\varpi)}\varrho^{n-\varpi-1} \sum_{s=0}^{\infty}\frac{a_s(n)_s}{(n-m)_s}\frac{{B_h}^{\Upsilon,\upsilon,k}(n-m+s, m-\varpi)}{B(n-m,m-\varpi)}\varrho^s. \quad (22)$$

Proof. Applying the theorem (2.1) and (2.2), we have:

$$C_{\varrho}^{\varpi}(\varrho^{n-1}f(\varrho), h, \Upsilon, \upsilon)$$

$$= \sum_{s=0}^{\infty} a_s C_{\varrho}^{\varpi}(\varrho^{n+s-1}, h), \quad (23)$$

$$= \frac{\Gamma(n)}{\Gamma(n-\varpi)}\varrho^{n-\varpi-1}\times \sum_{s=0}^{\infty}\frac{a_s(n)_s}{(n-\varpi)_s}\frac{{B_h}^{\Upsilon,\upsilon,k}(n-m+s, m-\varpi)}{B(n-m+s,m-\varpi)}\varrho^s, \quad (24)$$

$$= \frac{\Gamma(n)}{\Gamma(n-\varpi)}\varrho^{n-\varpi-1} \sum_{s=0}^{\infty}\frac{a_s(n)_s}{(n-m)_s}\frac{{B_h}^{\Upsilon,\upsilon,k}(n-m+s, m-\varpi)}{B(n-m,m-\varpi)}\varrho^s. \quad (25)$$

Hence the theorem is established.

Theorem 2.4. *The leading result hold true:*

$$C_{\varrho}^{n-\varpi}(\varrho^{n-1}(1-\varrho)^{-\xi},h,\Upsilon,\upsilon)=\frac{\Gamma(n)}{\Gamma(\varpi)}\varrho^{\varpi-1}$$

$$\sum_{s=0}^{\infty}\frac{(\xi)_s(n)_s}{(n-m)_s}\frac{B_h{}^{\Upsilon,\upsilon,k}(n-m+s,m-n+\varpi)}{B(n-m,\varpi-n+m)}\frac{z^s}{s!},\tag{26}$$

$$where\quad m-1<\Re(n-\varpi)<m<\Re(n)>0$$

and

$$\Re(h)>0,Re(\Upsilon),Re(\upsilon)>0.$$

Proof. Taking the power series expansion of $(1-\varrho)^{-\xi}$ in theorem (2.1), we have

$$C_{\varrho}^{n-\varpi}(\varrho^{n-1}(1-\varrho)^{-\xi},h,\Upsilon,\upsilon)$$

$$=C_{\varrho}^{n-\varpi}(\varrho^{n-1}\sum_{s=0}^{\infty}(\xi)_s\frac{(\varrho)^s}{s!},h,\Upsilon,\upsilon)\tag{27}$$

$$=\sum_{s=0}^{\infty}\frac{(\xi)_s}{s!}C_{\varrho}^{n-\varpi}(\varrho^{n+s-1},h,\Upsilon,\upsilon)\tag{28}$$

$$=\sum_{s=0}^{\infty}\frac{(\xi)_s}{s!}\frac{\Gamma(n+s)}{\Gamma(n+s-m)\Gamma(m-n+\varpi)}$$

$$B_h{}^{\Upsilon,\upsilon,k}(n-m+s,m-n+\varpi)\tag{29}$$

$$=\frac{\Gamma(n)}{\Gamma(\varpi)}\varrho^{\varpi-1}\sum_{s=0}^{\infty}\frac{(\xi)_s(n)_s}{(n-m)_s}$$

$$\frac{B_h{}^{\Upsilon,\upsilon,k}(n-m+s,m-n+\varpi)}{B(n-m,\varpi-n+m)}\frac{z^s}{s!},\tag{30}$$

which is the assumed result.

Theorem 2.5. *The leading result hold true*

$$C_{\varrho}^{n-\varpi}(\varrho^{n-1}(1-a\varrho)^{-\xi}(1-b\varrho)^{-\kappa},h,\Upsilon,\upsilon)$$

$$=\frac{\Gamma(n)}{\Gamma(\varpi)}\varrho^{\varpi-1}\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\xi)_s(\kappa)_p(n)_{s+p}}{(n-m)_{s+p}}\times$$

$$\frac{(a\varrho)^s}{s!}\frac{(b\varrho)^p}{p!}\frac{B_h{}^{\Upsilon,\upsilon,k}(n-m+s,\varpi-n)}{B(n-m,m+\varpi-n)},\tag{31}$$

where

$$m-1<\Re(n-\varpi)<m<\Re(n)>0$$

and

$$\Re(h)>0,Re(\Upsilon),Re(\upsilon)>0.$$

Proof. In order to prove the equation (31), we use power series

$$(1-a\varrho)^{-\xi}(1-b\varrho)^{-\kappa}=\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}(\xi)_s(\kappa)_p\frac{(a\varrho)^s}{s!}\frac{(b\varrho)^p}{p!}$$

Now applying theorem (2.4), we obtain the following:

$$C_{\varrho}^{n-\varpi}(\varrho^{n-1}(1-a\varrho)^{-\xi}(1-b\varrho)^{-\kappa},h,\Upsilon,\upsilon)$$

$$=\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}(-\xi)_n\kappa)_p\frac{(a)^s}{s!}\frac{(b)^p}{p!}C_{\varrho}^{n-\varpi}(\varrho^{n+s+p-1},h,\Upsilon,\upsilon).\tag{32}$$

By using theorem (2.1), we get the desired result.

Theorem 2.6. *The leading result holds true:*

$$C_{\varrho}^{\varpi}(e^{\varrho},h,\Upsilon,\upsilon)=\frac{\varrho^{m-\varpi}}{\Gamma(m-\varpi)}\times$$

$$\sum_{n=0}^{\infty}\frac{\varrho^n}{n!}B_h{}^{\Upsilon,\upsilon,k}(n+1,m-\varpi),\tag{33}$$

$$where\quad m-1<\Re(n-\varpi)<m<\Re(n)>0$$

and

$$\Re(h)>0,Re(\Upsilon),Re(\upsilon)>0.$$

Proof. By using series form of the exponential function and by using the theorem (2.2), we get desired result.

Theorem 2.7. *The leading result hold true*

$$C_{\varrho}^{\varpi}({}_2F_1(\chi,\kappa;\xi:\varrho),h,\Upsilon,\upsilon)$$

$$=\frac{(\chi)_m(\kappa)_m}{(\xi)_m}\frac{\varrho^{m-\varpi}}{\Gamma(1-\varpi+m)}\sum_{n=0}^{\infty}\frac{(\chi+m)_n(\kappa+m)_n}{(\xi)_{n+m}(1-\varpi+m)_n}$$

$$\frac{B_h{}^{\Upsilon,\upsilon,k}(n+1,m-\varpi)}{B(n+1,m-\varpi)}\varrho^n,\tag{34}$$

where $m-1<\Re(\varpi)<m$ *for* $\chi,\kappa,\xi\in C$ *st* $R(\chi)>0,R(\xi)>R(\kappa)>0$ *and* $|\varrho|<1,\Re(h)>0,Re(\Upsilon),Re(\upsilon)>0.$

Proof. As per this equation

$$C_{\varrho}^{\varpi}({}_2F_1(\chi,\kappa;\xi:\varrho),h,\Upsilon,\upsilon)=$$

$$\sum_{n=0}^{\infty}\frac{(\chi)_n(\kappa)_n}{(\xi)_n n!}C_{\varrho}^{\varpi}(\varrho^n,h,\Upsilon,\upsilon).\tag{35}$$

And applying the theorem (2.1) in the above, we obtained what we want.

Theorem 2.8. *The leading result hold true*

$$C_\varrho^{\varpi}(E_{\xi,\kappa}^{\chi}(\varrho),h,\Upsilon,\upsilon)=$$

$$\frac{\varrho^{m-\varpi}\chi_m}{\Gamma(m-\varpi)}\sum_{n=0}^{\infty}\frac{(\chi+m)_n}{\Gamma(\xi n+\xi m+\kappa)}\frac{\varrho^n}{n!},\tag{36}$$

where

$$m-1<\Re(\varpi)<m$$

and

$$\Re(h)>0, Re(\Upsilon), Re(\upsilon)>0.$$

Here, $E^{\lambda}_{\alpha,\beta}$ *is generalised Mittag-Leffler function which was introduced by Prabhakar, defined as:*

$$E_{\alpha,\beta}^{\lambda}(z)=\sum_{n=0}^{\infty}\frac{(\lambda)_n z^n}{\Gamma(\alpha n+\beta)n!},\tag{37}$$

where $\alpha, \beta, \lambda \in C$, $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\lambda) > 0$ *and* $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$ *denotes a pochhammer symbol.*

Proof. Applying theorem (2.2) in L.H.S, we get

$$C_\varrho^{\varpi}(E_{\xi,\kappa}^{\chi}(\varrho),h,\Upsilon,\upsilon)=\sum_{n=0}^{\infty}\frac{(\chi)_n}{\Gamma(\xi n+\kappa)n!}C_\varrho^{\varpi}(\varrho^n,h,\Upsilon,\upsilon),\tag{38}$$

using the theorem (2.1) leads us to the following solution.

Theorem 2.9. *The leading result hold true*

$$C_\varrho^{\varpi}\left({}_p\Psi_q\left[\begin{matrix}(\lambda_i,\eta_i)_{1,p}\\(\mu_i,\zeta_i)_{1,q}\end{matrix};z\right],h,\Upsilon,\upsilon\right)=\frac{\varrho^{m-\varpi}}{\Gamma(m-\varpi)}\times$$

$$\sum_{n=0}^{\infty}\frac{\Pi_{i=1}^{m}\Gamma(\lambda_i+\eta_i n)}{\Pi_{j=1}^{n}\Gamma(\mu_i+\zeta_i n)}B_h^{\Upsilon,\upsilon,k}(n+1,m-\varpi)\frac{\varrho^n}{n!},\tag{39}$$

$$where\Re(h)>0, Re(\Upsilon), Re(\upsilon)>0$$

${}_p\Psi_q$ *Denote Fox wright function, which is defined in paper Kiymaz et al. (2016) as:*

$${}_p\Psi_q\left[\begin{matrix}(\lambda_i,\eta_i)_{1,p}\\(\mu_i,\zeta_i)_{1,q}\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{\Pi_{i=1}^{m}\Gamma(\lambda_i+\eta_i n)}{\Pi_{j=1}^{n}\Gamma(\mu_i+\zeta_i n)}\frac{\varrho^n}{n!}$$

and the coefficients $\eta_i (i = 1, \ldots\ldots, p)$ *and* ζ_j $(j =1, \ldots\ldots\ldots, q)$ *are positive real numbers, such that*

$$1+\sum_{j=1}^{q}\zeta_j-\sum_{i=1}^{p}\eta_i\geq 0.$$

Proof. Using the theorem (2.2) in L.H.S, we get:

$$C_\varrho^{\varpi}\left({}_p\Psi_q\left[\begin{matrix}(\lambda_i,\eta_i)_{1,p}\\(\mu_i,\zeta_i)_{1,q}\end{matrix};z\right],h,\Upsilon,\upsilon\right)$$

$$=C_\varrho^{\varpi}\left\{\sum_{n=0}^{\infty}\frac{\Pi_{i=1}^{m}\Gamma(\lambda_i+\eta_i n)\varrho^n}{\Pi_{j=1}^{n}\Gamma(\mu_i+\zeta_i n)n!}:h,\Upsilon,\upsilon\right\}.\tag{40}$$

By applying the theorem (2.3).

Again using the theorem (2.1) in the above equation, hence the theorem is proved.

Theorem 2.10. *The leading Mellin Transformation formula hold true:*

$$M(C_\varrho^{\varpi}(\varrho^n,h,\Upsilon,\upsilon);h\to r)$$

$$=\frac{\Gamma(n+1)\Gamma(r)}{\Gamma(n-m+1)\Gamma(m-\varpi)\Gamma(\upsilon-r)}\times$$

$$\frac{\pi}{sin(\pi\upsilon)}\varrho^{n-\varpi}B(n-m+kr+1,m-\varpi+kr),\tag{41}$$

$$when\quad \Upsilon=1\quad where\quad \Re(n)>m-1\quad \Re(r)>0.$$

Proof. Using the definition of the Mellin transformation on equation (23) and theorem (2.1) in L.H.S, we get:

$$M(C_\varrho^{\varpi}(\varrho^n,h,\Upsilon,\upsilon);h\to r)$$

$$=\int_0^{\infty}h^{r-1}C_\varrho^{\varpi}(\varrho^n,h,\Upsilon,\upsilon)dh,\tag{42}$$

futher using equation (14), we get:

$$M(C_\varrho^{\varpi}(\varrho^n,h,\Upsilon,\upsilon);h\to r)$$

$$=\int_0^{\infty}h^{r-1}\left\{\frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m-\varpi)}\times\right.$$

$$\left.\int_0^{\varrho}(\varrho-t)^{m-\varpi-1}{}_1F_1\left(\Upsilon;\upsilon;\frac{-h\varrho^{2k}}{t^k(\varrho-t)^k}\right)\right\}t^{n-m}dtdh,\tag{43}$$

and then substituting $t = u\varrho$, we get

$$M(C_{\varrho}^{\varpi}(\varrho^n, h, \Upsilon, v); h \to r)$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m-\varpi)} \varrho^{n-\varpi} \times \int_0^\infty \left\{ h^{r-1} \int_0^1 u^{n-m}(1-u)^{m-\varpi-1} \times {}_1F_1\big(\Upsilon; v; \frac{-h}{u^k(1-u)^k}\big) du \right\} dh, \tag{44}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m-\varpi)} \varrho^{n-\varpi} \times \int_0^1 \left\{ u^{n-m}(1-u)^{m-\varpi-1} \int_0^\infty h^{r-1} \times {}_1F_1\big(\Upsilon; v; \frac{-h}{u^k(1-u)^k}\big) dh \right\} du, \tag{45}$$

further substituting $\nu = \frac{h}{u^k(1-u)^k}$, we get

$$= \frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m-\varpi)} \varrho^{n-\varpi} \times \int_0^1 \left\{ u^{n-m+kr}(1-u)^{m-\varpi+kr-1} du \times \int_0^\infty \nu^{r-1} {}_1F_1\big(\Upsilon; v; -\nu\big) d\nu \right\}. \tag{46}$$

By using the result from (Gradshteyn and Ryzhik (2014), page (821)), we get

$$= \frac{\Gamma(n+1)\Gamma(v)\Gamma(\Upsilon-v)\Gamma(r)}{\Gamma(n-m+1)\Gamma(m-\varpi)\Gamma(v-r)} \times \varrho^{n-\varpi} B(n-m+kr+1, m-\varpi+kr), \tag{47}$$

when $\Upsilon = 1$, we get

$$= \frac{\Gamma(n+1)\Gamma(r)}{\Gamma(n-m+1)\Gamma(m-\varpi)\Gamma(v-r)} \times \frac{\pi}{sin(\pi v)} \varrho^{n-\varpi} B(n-m+kr+1, m-\varpi+kr). \tag{48}$$

Hence, this becomes the desired result.

Conclusion

This paper mainly deals with the extension of Caputo fractional derivative operator using the Confluent hypergeometric function, as its kernal. Many results are derived after applying on various special known functions. After thorough investigation, we reach at this conclusion that, when we assume ζ = η, in all the above results mentioned in this paper, they will be associated with the Caputo extended fractional derivative operator defined by Kiymazn *et.al.* (2016). If h=0 and ζ = η, then all the results demonstrated in this paper will be valid for the Classical Caputo fractional derivative operator function which is defined in Samko *et.al.* (1993).

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